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PARTIAL DERIVATIVES OF MATRICES REPRESENTING RIGID BODY ROTATIONS

Mathematical Physics Branch

MISSION PLANNING AND ANALYSIS DIVISION

MANNED SPACECRAFT CENTER
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SUMMARY AND INTRODUCTION

The method presented for computing the partial derivatives of matrices representing rigid body rotations is applied here to the formulation of an RTCC processor presently under development for the lunar landing mission. This processor will determine the lunar module attitude with respect to the local vertical coordinate system by processing rendezvous radar shaft and trunnion angles with a weighted least squares filter. As will be seen, the necessary partials in final form are rather simple, but derivation would be cumbersome without the method derived here.

In fact, to derive the required partials for the above mentioned processor with the standard method would require inversion of three 3-by-3 matrices, the partial derivative of a 3-by-3 matrix with respect to its rotation angle, the multiplications of six 3-by-3 matrices, and a vector dot product.

With the method given here, the derivation requires the definition of two unit vectors, a multiplication of one 3-by-3 matrix by a vector, and a dot product.

The procedure described in this note can have wide application in all areas of dynamics involving rigid body rotations both inside and outside the realm of space technology. Examples include solving for tracking station location errors by processing angles, determining the final position of a rigid body after undergoing a finite rotation, determining inertial platform alignment errors with angle data, and determining any angular state which is a function of some angular observation.

RIGID ROTATION MATRICES IN THREE SPACE

The differential equation for a rotation of a vector $\hat{R}$, with respect to the variable σ, about a unit vector, $\hat{N}$, is given in any text on physics as

$$\frac{\partial \hat{R}}{\partial \sigma} = \hat{N} \times \hat{R} \tag{1}$$

The integration of this equation for a finite σ will represent the rigid rotation of the vector $\hat{R}$ about the fixed unit vector $\hat{N}$ through the angle σ.

The vector cross product, in three space, is equivalent to a matrix transformation. If we take

$$\left.\begin{aligned} \hat{N} &= \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}, \quad |\hat{N}| = 1 \\ \hat{R} &= \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \end{aligned}\right\} \tag{2}$$

then

$$\hat{N} \times \hat{R} = \begin{bmatrix} n_2\, r_3 - n_3\, r_2 \\ n_3\, r_1 - n_1\, r_3 \\ n_1\, r_2 - n_2\, r_1 \end{bmatrix}$$

This may also be written as

$$\hat{N} \times \hat{R} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{3}$$

If we write

$$\hat{N} \times \hat{R} = A(\hat{N})R \tag{3a}$$

then we can define the notation

$$N \times = A(\hat{N}) \tag{4}$$

where the matrix A is given by

$$A(\hat{N}) = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \tag{4a}$$

Using this notation, the differential equation for the rotating vector may be written as

$$\frac{\partial \hat{R}}{\partial \sigma} = A(\hat{N})\hat{R} \tag{5}$$

This is a linear differential equation, and a solution may be obtained in terms of the unit characteristic solution

$$\hat{R} = C(\hat{N}, \sigma)\ \hat{R}_o \tag{6}$$

where the matrix $C(\hat{N}, \sigma)$ satisfies the following conditions:

$$\frac{\partial C(\hat{N}, \sigma)}{\partial \sigma} = A(\hat{N})\ C(\hat{N}, \sigma) \tag{7}$$

with the initial conditions given by the unit matrix in three space, i.e.,

$$C(\hat{N}, 0) = I_3 \tag{7a}$$

The solution of equation (7) is given by

$$C(\hat{N}, \sigma) = e^{\sigma A(\hat{N})} I_3 \tag{8}$$

Expanding the exponential matrix function, we have

$$C(\hat{N}, \sigma) = I_3 + \sigma A(\hat{N}) + \frac{\sigma^2}{2!} A^2(\hat{N}) + \ldots + \frac{\sigma^k}{k!} A^k(\hat{N}) \ \ldots \tag{9}$$

It can be shown that the matrix $A(\hat{N})$ satisfies the following characteristic matrix equation

$$A^3(\hat{N}) = -A(\hat{N}) \tag{10}$$

The solution of $C(\hat{N}, \sigma)$ may be written as

$$C(\hat{N}, \sigma) = I_3 + (1 - \cos\sigma)A^2(\hat{N}) + \sin\sigma\, A(\hat{N}) \tag{11}$$

This may also be written as

$$C(\hat{N}, \sigma) = I_3 + (1 - \cos\sigma)\ (\hat{N}\times)\ (\hat{N}\times) + \sin\sigma\ \hat{N}\times \tag{11a}$$

The inverse of $C(\hat{N}, \sigma)$ is given by changing σ to $-\sigma$.

$$C^{-1}(\hat{N}, \sigma) = C(\hat{N}, -\sigma)$$

It can be easily shown that this matrix is invertable by the following discussion:

From equation (8),

$$C(\hat{N}, \sigma) = e^{\sigma A(\hat{N})} = I_3 + \sigma A(\hat{N}) + \frac{[\sigma A(\hat{N})]^2}{2!} + \ldots + \frac{[\sigma A(\hat{N})^n]}{n!} \tag{8a}$$

If we write $A(\hat{N})$ in the classical diagonal canonical form

$$T^{-1} A(\hat{N})T = \Lambda$$

or

$$A(\hat{N}) = T\ \Lambda\ T^{-1}$$

then $[A(\hat{N})]^2 = (T\ \Lambda\ T^{-1})\ (T\ \Lambda\ T^{-1}) = T\ \Lambda^2\ T^{-1}$

In general $\quad [A(N)]^n = T\ \Lambda^n\ T^{-1}$

We now rewrite equation (8a)

$$C(\hat{N}, \sigma) = e^{\sigma\ T\ \Lambda\ T^{-1}} = I + \sigma(T\ \Lambda\ T^{-1}) + \frac{\sigma\ T\ \Lambda^2 T^{-1}}{2!} + \ldots + \frac{\sigma\ T\ \Lambda^n T^{-1}}{n!} \tag{8b}$$

Pre-and postmultiplying by T^{-1} and T, respectively

$$T^{-1}\ C(\hat{N}, \sigma)\ T = I + \sigma\Lambda + \frac{\sigma\Lambda^2}{2!} + \ldots + \frac{\sigma\Lambda^n}{n!}$$

or

$$T^{-1}\ C(\hat{N}, \sigma)T = e^{\sigma\Lambda}$$

Now this is the classical canonical form of the matrix $C(\hat{N}, \sigma)$. Therefore, the eigenvalues of $C(\hat{N}, \sigma)$ are $e^{\sigma\Lambda}$, where Λ are the eigenvalues of $A(\hat{N})$. It is easily seen that the eigenvalues of $A(\hat{N})$ are 0, $\pm$ i. [see equation (10)]

The determinant of $C(\hat{N}, \sigma)$ is then $(e^{0})\ (e^{i\sigma})\ (e^{-i\sigma}) = 1$ and the matrix is invertable.

We now prove that $C(\hat{N}, \sigma)$ is an orthogonal transformation. We have

$$\frac{\partial C^T}{\partial \sigma} = C^T\ A^T \tag{12}$$

But A^T is skew symmetric and $A^T = -A$ where the superscript T denotes the transpose. It follows that

$$\frac{\partial C^T}{\partial \sigma} = -C^T\ A \tag{13}$$

We also have

$$C^{-1}\ C = I \tag{14}$$

The derivative of equation (14) yields

$$\begin{aligned} \frac{\partial}{\partial \sigma} C^{-1}\ C &= -C^{-1}\ \frac{\partial C}{\partial \sigma} \\ &= -C^{-1}\ AC \end{aligned} \tag{15}$$

or

$$\frac{\partial}{\partial \sigma} C^{-1} = C^{-1}\ A \tag{16}$$

It follows that

$$C^{-1} = C^T \tag{17}$$

which proves that C is orthogonal.

An interesting and useful consequence of equation (7) is

$$\frac{\partial C(\hat{N}, \sigma)}{\partial \sigma}\ C^{-1}\ (\hat{N}, \sigma) = A(\hat{N}) = \hat{N}\ \times \tag{18}$$

Moreover, if we have a product of two transformations, such as,

$$E = D(\hat{M}, \beta)\ C(\hat{N}, \sigma) \tag{19}$$

the partial derivative of E with respect to the generic variable γ is given by

$$\frac{\partial E}{\partial \gamma} = \frac{\partial D(\hat{M}, \beta)}{\partial \gamma} C(\hat{N}, \sigma) + D(\hat{M}, \beta) \frac{\partial C(\hat{N}, \sigma)}{\partial \gamma} \tag{20}$$

Postmultiplying with the transpose of E, we have

$$\frac{\partial E}{\partial \gamma} E^T = \frac{\partial \beta}{\partial \gamma} B(\hat{M}) + \frac{\partial \sigma}{\partial \gamma} D(\hat{M}, \beta) A(\hat{N}) D^T (\hat{M}, \beta) \tag{21}$$

where $A(\hat{M}) = \hat{M} \times$ (21a)

We now prove that

$$D(\hat{M}, \beta) A(\hat{N}) D^T (\hat{M}, \beta) = A(\hat{L}) = \hat{L} \times \tag{22}$$

where

$$\hat{L} = D(\hat{M}, \beta) \hat{N} \tag{22a}$$

The triple matrix product DAD^T is given by

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \begin{bmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \end{bmatrix} \tag{23}$$

Carrying out the multiplication for the first row of B we have

$$\left.\begin{aligned} b_{11} &= n_1(d_{12}\, d_{13} - d_{12}\, d_{13}) + n_2(d_{11}\, d_{13} - d_{11}\, d_{13}) \\ &\quad + n_3(d_{11}\, d_{12} - d_{11}\, d_{12}) \\ b_{12} &= n_1(d_{13}\, d_{22} - d_{13}\, d_{23}) + n_2(d_{11}\, d_{23} - d_{21}\, d_{13}) \\ &\quad + n_3(d_{12}\, d_{21} - d_{11}\, d_{22}) \\ b_{13} &= n_1(d_{13}\, d_{32} - d_{12}\, d_{33}) + n_2(d_{11}\, d_{33} - d_{13}\, d_{31}) \\ &\quad + n_3(d_{12}\, d_{31} - d_{11}\, d_{32}) \end{aligned}\right\} \tag{24}$$

Since the matrix $D(\hat{M}, \beta)$ is orthogonal we have that

$$\left.\begin{aligned}
d_{31} &= d_{12}\, d_{23} - d_{13}\, d_{22} \\
d_{32} &= d_{21}\, d_{13} - d_{11}\, d_{23} \\
d_{33} &= d_{11}\, d_{22} - d_{12}\, d_{21} \\
d_{21} &= d_{13}\, d_{32} - d_{12}\, d_{33} \\
d_{22} &= d_{11}\, d_{33} - d_{13}\, d_{31} \\
d_{23} &= d_{12}\, d_{31} - d_{11}\, d_{32}
\end{aligned}\right\} \qquad (25)$$

Consequently,

$$\left.\begin{aligned}
b_{11} &= 0 \\
b_{12} &= -d_{31}\, n_1 - d_{32}\, n_2 - d_{33}\, n_3 = -\ell_3 \\
b_{13} &= d_{21}\, n_1 + d_{22}\, n_2 + d_{23}\, n_3 = \ell_2
\end{aligned}\right\} \qquad (26)$$

Similarly we may show

$$\left.\begin{aligned}
b_{21} &= \ell_3 \\
b_{22} &= 0 \\
b_{23} &= -\ell_1 \\
b_{31} &= -\ell_2 \\
b_{32} &= \ell_1 \\
b_{33} &= 0
\end{aligned}\right\} \qquad (27)$$

It follows that

$$D(\hat{M}, \beta)\, A(\hat{N})\, D^T(\hat{M}, \beta) = \hat{L} \times \qquad (28)$$

where

$$D(\hat{M}, \beta)\, \hat{N} = \hat{L} \qquad (28a)$$

The final result for the product transformation is given by

$$\frac{\partial E}{\partial \gamma} E^T = \frac{\partial \beta}{\partial \gamma} \hat{M} \times + \frac{\partial \sigma}{\partial \gamma} \hat{L} \times \tag{29}$$

APPLICATION

The utility of this formulation may be seen in the discussion below which derives the partial derivatives to be used in an Apollo RTCC processor.

Just after the lunar landing, the orientation of the LM is recorded with respect to a mean fixed coordinate system. If this stored alignment changes (due to LM settling) and realignment cannot be made with the alignment optical telescope; a separate method for LM altitude determination must be available. The reference presents a method in which the LM body attitude with respect to the local vertical coordinate system is determined by processing rendezvous radar (RR) shaft and trunnion angles with a weighted least squares filter. This method requires the partial derivatives of the observations with respect to a set of state elements. This state will be defined as three positive rotations about the LM local vertical system axes. The three rotations are ordered as follows:

1. About the local vertical, α_1.
2. About the displaced east, α_2.
3. About the displaced north, α_3.

The transformation from local vertical to LM body coordinates is then defined:

$$\hat{R}_B = \begin{bmatrix} \cos\alpha_3 & \sin\alpha_3 & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha_2 & 0 & -\sin\alpha_2 \\ 0 & 1 & 0 \\ \sin\alpha_2 & 0 & \cos\alpha_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \hat{R}_{LV} \tag{30}$$

where $\hat{R}_B$ is a vector in the body coordinates and $\hat{R}_{LV}$ is the same vector in the local vertical system.

From the body vector the observations may be written as

$$\left.\begin{aligned} \sin(T) &= -Y_B \\ \tan(S) &= \frac{X_B}{Z_B} \end{aligned}\right\} \tag{30}$$

where S and T are RR shaft and trunnion angles.

Given the above definition, the least squares method requires the following partial derivatives

$$\frac{\partial O_B}{\partial \alpha_i} \tag{31}$$

where O_B is S or T and i = 1, 2, 3.

This may be written as

$$\frac{\partial O_B}{\partial \alpha_i} = \frac{\partial O_B}{\partial \bar{R}_B} \cdot \frac{\partial \bar{R}_B}{\partial \alpha_i} \tag{32}$$

The partials of O_B with respect to $\bar{R}_B$ follow from the definition in equation (31). They are

$$\frac{\partial S}{\partial \hat{R}_B} = \frac{1}{\cos(T)} \quad [\cos S,\ 0,\ -\sin S] \tag{33}$$

$$\frac{\partial T}{\partial \hat{R}_B} = [-\sin T \sin S,\ -\cos T,\ -\sin T \cos S] \tag{34}$$

Now by application of equation (5),

$$\frac{\partial \hat{R}_B}{\partial \alpha_i} = A(\hat{N}_1)\ \hat{R}_B \tag{35}$$

Here $\hat{N}_i$ is a unit vector about which $\hat{R}_B$ is rotated through α_i (i = 1, 2, 3).

Inspection of the vector columns in the matrices of equation (30) define the N_i's. They are

$$\hat{N}_1 = [-\cos\alpha_2 \cos\alpha_3,\ \sin\alpha_3 \cos\alpha_2,\ -\sin\alpha_2]$$

$$\hat{N}_2 = [-\sin\alpha_3,\ -\cos\alpha_3,\ 0] \qquad (36)$$

$$\hat{N}_3 = [0,\ 0,\ -1]$$

From the geometry of the LM body axes, $\hat{R}_B$ may be written as:

$$\hat{R}_B = \begin{bmatrix} \cos T \sin S \\ -\sin T \\ \cos T \cos S \end{bmatrix} \qquad (37)$$

using the definition (equation 4a) for $A(N_i)$ and applying equations (36) and (37) to equation (35)

$$\frac{\partial \hat{R}_B}{\partial \alpha_1} = A(\hat{N}_1)\,\hat{R}_B = \begin{bmatrix} -\sin\alpha_2 \sin T + \sin\alpha_3 \cos\alpha_2 \cos T \cos S \\ -\sin\alpha_2 \cos T \sin S + \cos\alpha_2 \cos\alpha_3 \cos T \cos S \\ -\sin\alpha_3 \cos\alpha_2 \cos T \sin S + \cos\alpha_2 \cos\alpha_3 \sin T \end{bmatrix} \qquad (38)$$

$$\frac{\partial \hat{R}_B}{\partial \alpha_2} = A(\hat{N}_2)\,\hat{R}_B = \begin{bmatrix} -\cos\alpha_3 \cos T \cos S \\ \sin\alpha_3 \cos T \cos S \\ \cos\alpha_3 \cos T \sin S + \sin\alpha_3 \sin T \end{bmatrix} \qquad (39)$$

$$\frac{\partial \hat{R}_B}{\partial \alpha_3} = A(\hat{N}_3)\,\hat{R}_B = \begin{bmatrix} -\sin T \\ -\cos T \sin S \\ 0 \end{bmatrix} \qquad (40)$$

The dot products of these vectors (equations 38, 39, and 40) with the vectors (equations 33 and 34) as defined by equation (32) forms the required six partial derivatives.

$$\frac{\partial S}{\partial \sigma_1} = \frac{\partial S}{\partial \hat{R}_B} \cdot \frac{\partial \hat{R}_B}{\partial \alpha_1} = \sin \alpha_3 \cos \alpha_2 - \sin \alpha_2 \cos S \tan T - \cos \alpha_2 \cos \alpha_3 \sin S \tan T \tag{41}$$

$$\frac{\partial S}{\partial \alpha_2} = \frac{\partial S}{\partial \hat{R}_B} \cdot \frac{\partial \hat{R}_B}{\partial \alpha_2} = -\cos \alpha_3 - \sin S \sin \alpha_3 \tan T \tag{42}$$

$$\frac{\partial S}{\partial \alpha_3} = \frac{\partial S}{\partial \hat{R}_B} \cdot \frac{\partial \hat{R}_B}{\partial \alpha_3} = -\tan T \cos S \tag{43}$$

$$\frac{\partial T}{\partial \alpha_1} = \frac{\partial T}{\partial \hat{R}_B} \cdot \frac{\partial \hat{R}_B}{\partial \alpha_1} = \sin \alpha_2 \sin S - \cos \alpha_2 \cos \alpha_3 \cos S \tag{44}$$

$$\frac{\partial T}{\partial \alpha_2} = \frac{\partial T}{\partial \hat{R}_B} \cdot \frac{\partial \hat{R}_B}{\partial \alpha_2} = -\sin \alpha_3 \cos S \tag{45}$$

$$\frac{\partial T}{\partial \alpha_3} = \frac{\partial T}{\partial \hat{R}_B} \cdot \frac{\partial \hat{R}_B}{\partial \alpha_3} = \sin S \tag{46}$$

REFERENCES

1. Cockrell, B. F.: RTCC Requirements for Mission G: Lunar Module Attitude Determination Using Onboard Observation. MSC IN 68-FM-125, May 28, 1968.